Slowly Oscillating Lifting Surfaces at Subsonic and Supersonic Speeds

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The paper presents an unsteady aerodynamic influence coefficient method based on the low-frequency approximation. The influence coefficients are of a type which have been used to compute steady flow about wing-body combinations; therefore, the new method may be extended readily to low-frequency unsteady flow about wing-body combinations. The validity of the method is demonstrated by comparisons with numerical results from conventional, unsteady lifting surface methods. The method is valid for arbitrary wings in supersonic flow and for wings of finite span in subsonic flow. The method, when extended to include wing-body-tail interactions, will have important applications for predicting stability, control, and gust response characteristics of large airplanes. Dynamic stability derivatives and pressure distributions are given for several planforms. The comparison with either analytical or other well established numerical methods shows good agreement.

Nomenclature

a_{ij} = unsteady aerodynamic influence coefficient
A_{ij} = steady aerodynamic influence coefficient
AR = aspect ratio
Δc_p = unsteady pressure jump, referred to $\frac{1}{2}\rho_\infty U_\infty^2$
ΔC_p = steady pressure jump
c_L = lift coefficient
c_M = pitching moment coefficient
c_{ref} = reference length
H = unit step function
i = $(-1)^{1/2}$, imaginary unit
I_{ij} = integrated downwash
k = $\omega c_{ref}/U_\infty$, reduced frequency
K = kernel function
M_∞ = freestream Mach number
S = wing area
S_j = panel area
S_{ref} = reference area
U_∞ = freestream velocity
w = unsteady downwash
W = steady downwash
x,y,z = nondimensional Cartesian coordinates
x_0 = reference axis of pitching moment
x_c = x-location of panel centroid
α = angle of attack
β = $|1 - M_\infty^2|^{1/2}$
ω = circular frequency
ϕ = unsteady velocity potential, referred to $U_\infty c_{ref}$
Φ = steady velocity potential
ρ_∞ = freestream density
[ ] = square matrix
⌊ ⌋ = row matrix
{ } = column matrix

Subscripts and Superscripts

i,j = influenced and influencing panel
LE,TE = leading edge, trailing edge
qs = quasisteady
s = steady
R = real part
I = coefficient of imaginary part
(0) = panel with constant steady load
(1) = panel with in x-linearly varying load
$\hat{}$ = complex amplitude

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Introduction

MANY methods have been developed for predicting the stability derivatives of airplanes using aerodynamic influence coefficients in one form or another. The method described by Roskam and Dusto[1] uses the steady aerodynamic influence coefficients derived by Woodward.[2] These relate the aerodynamic load on a small element of the airplane's surface (a panel) to the flow incidence at another panel.

This paper describes an extension of Woodward's steady influence coefficients to low frequency unsteady flow. These new influence coefficients are based on the low frequency approximation to unsteady flow theory from Miles[3] assuming that the reduced frequency of the airplane's motion is small by comparison with unity. The method has rather broad application in supersonic flow; but as shown by Brune,[4] its validity is restricted to wings of finite aspect ratio in subsonic flow.

The low frequency influence coefficients are obtained by applying a reduction formula derived by Brune.[5] In this formula, the boundary value problem for harmonically oscillating lifting surfaces is completely reduced to a sequence of steady flow problems. Similar reduction formulas have been used previously by Miles[3,6] and Tobak and Lessing[7] to solve unsteady flow problems. Göthert and Otto[8] showed that a close relation exists between Ref. 7 and the Multhopp–Garner[9] theory.

The characteristic frequencies of the rigid body motion of large aircraft lie within the low frequency approximation and the first several structural modes also satisfy the criterion. Thus, the method described herein can be expected to predict adequately the aerodynamic damping and inertia of these motions. It can be readily applied to the analytical method of Dusto[10] for predicting stability characteristics of large flexible airplanes.

The dynamic stability derivatives will be shown to be independent of frequency. The equations of motion are, therefore, formulated with constant coefficients and the characteristics of damped motions are evaluated directly. The stability derivatives are valid for arbitrary motions with the restriction that they must map into the frequency domain in the region where $k \ll 1$.

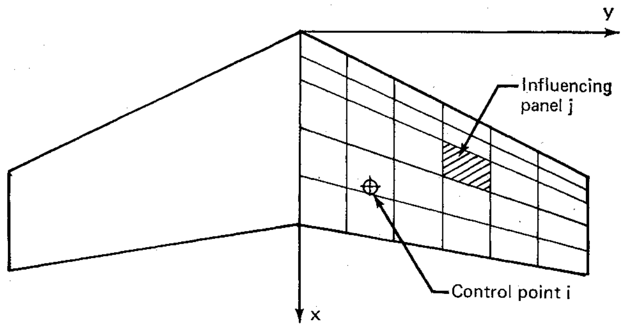

Fig. 1 Typical panel scheme.

Unsteady Aerodynamic Theory

Consider a thin wing in a uniform subsonic or supersonic freestream. Each element of the wing's surface is slowly oscillating perpendicular to the freestream direction. Linear inviscid theory will be assumed. Since harmonic oscillations shall be considered, the quantity of interest is the complex amplitude of the aerodynamic load $\Delta\hat{c}_p$. Letting the thin wing be represented by a distribution of unsteady vorticity in the x, y plane (Fig. 1), a well-known singular integral equation relates the complex amplitudes of downwash $\hat{w}$ and load $\Delta\hat{c}_p$.

$$\hat{w}(x,y) = \iint_S \Delta\hat{c}_p(\xi,\eta)K(x-\xi, y-\eta; k, M_\infty)d\xi d\eta \quad (1)$$

The kernel function K is a complicated expression which may be simplified for slowly oscillating wings. To be more specific, a slow oscillation is characterized by reduced frequencies k that are small compared to unity, $k \ll 1$. Watkins, Runyan, and Woolston[11] expanded the subsonic kernel function for small k values. The result valid to first order in frequency is

$$8\pi K(x-\xi, y-\eta) = \frac{1-ik(x-\xi)}{(y-\eta)^2}\left\{1+\frac{x-\xi}{[(x-\xi)^2+\beta^2(y-\eta)^2]^{1/2}}\right\} - \frac{ik}{[(x-\xi)^2+\beta^2(y-\eta)^2]^{1/2}} + O(k^2, k^2 \ln k) \quad (1a)$$

The equivalent form for the supersonic kernel function has been given by Watkins and Berman[12] as

$$4\pi K(x-\xi, y-\eta) = \frac{1-ik(x-\xi)}{(y-\eta)^2}H[(x-\xi)-\beta|y-\eta|]\times \frac{x-\xi}{[(x-\xi)^2-\beta^2(y-\eta)^2]^{1/2}} - \frac{ikH[(x-\xi)-\beta|y-\eta|]}{[(x-\xi)^2-\beta^2(y-\eta)^2]^{1/2}} + O(k^2) \quad (1b)$$

where H is the unit step function.

To solve Eq. (1) the wing planform is divided into a large number N of quadrilateral panels and one control point is associated with each panel. A typical panel scheme is shown in Fig. 1. An aerodynamic influence coefficient is now defined as the complex amplitude of the downwash at the control point i induced by panel j having a constant and purely real load.

In general, for $k = 0(1)$, an unsteady aerodynamic influence coefficient is given by the double integral over the panel area S_j

$$\hat{a}_{ij} = \iint_{S_j} K(x_i-\xi, y_i-\eta; k, M_\infty)d\xi d\eta \quad (2)$$

Superimposing the effects of all panels and requiring that the boundary condition of tangential flow on the surface is satisfied at each control point, the unsteady flow problem takes the form

$$[\hat{a}_{ij}]\{\Delta\hat{c}_{pi}\} = \{\hat{w}_i\} \quad (3)$$

The unknown oscillatory load is complex; but, when expanded in terms of reduced frequency, the load, valid to first order in frequency, is

$$\{\Delta\hat{c}_{pi}\} = \{\Delta c_{pRi}\} + ik\{\Delta c_{pIi}\} \quad (4)$$

where both Δc_{pR} and Δc_{pI} are independent of frequency. Similar first order in frequency expansions of AIC and downwash are assumed, i.e.,

$$\hat{a}_{ij} = a_{Rij} + ika_{Iij} \qquad \hat{w}_i = w_{Ri} + ikw_{Ii}$$

Hence, the solution‡ of Eq. (3) for $\Delta\hat{c}_p$ is

$$\{\Delta\hat{c}_{pi}\} = [a_{Rij}]^{-1}\{w_{Ri}\} + ik[a_{Rij}]^{-1}\{\bar{w}_i\} \quad (5)$$

with

$$\{\bar{w}_i\} = \{w_{Ii}\} - [a_{Iij}][a_{Rij}]^{-1}\{w_{Ri}\}$$

The second term of the modified downwash $\{\bar{w}_i\}$ represents the unsteadiness of the problem. If that term is neglected the approach is called quasisteady and the corresponding load is simply given by

$$\{\Delta\hat{c}_{pi}\}_{qs} = [a_{Rij}]^{-1}\{w_{Ri}\} + ik[a_{Rij}]^{-1}\{w_{Ii}\} \quad (6)$$

The outlined low-frequency theory can be applied to subsonic and supersonic flow problems. It has already been shown[4] that the approximation is valid for arbitrary wings in supersonic flow and wings of finite span in subsonic flow.

Aerodynamic Influence Coefficients

A previously published[5] reduction formula for the low frequency unsteady lifting surface problem can be applied to evaluate the double integral Eq. (2). By means of that reduction technique, low frequency aerodynamic influence coefficients can be expressed by a sequence of steady influence coefficients.

The formula for the complex amplitude of the unsteady velocity potential has been shown by reference[5] to be

$$\hat{\phi} = \Phi^{(0)} + ik\left(\Phi^{(1)} + \frac{M_\infty^2}{1-M_\infty^2}x\Phi^{(0)} + \frac{1}{M_\infty^2-1}\int_{-\infty}^{x}\Phi^{(0)}d\xi\right) \quad (7)$$

where the potentials $\Phi^{(0)}$ and $\Phi^{(1)}$ are solutions to aerodynamic problems of the type encountered in steady flow. The corresponding steady downwashes are

$$W^{(0)} = w_R \quad (8)$$

$$W^{(1)} = w_I + \frac{M_\infty^2}{M_\infty^2-1}xw_R + \frac{1}{1-M_\infty^2}\int_{-\infty}^{x} w_R d\xi \quad (9)$$

where w_R and w_I are the components of the unsteady downwash

$$\hat{w} = w_R + ikw_I$$

‡ It should be noticed that Eq. (5) is one possible formulation of the frequency expansion of an integral equation given by Miles[3] in Chap. 4 of his monograph. The real part of the AIC matrix can be identified as the operator M_0 in his Eq. (4.4.29).

The equivalent reduction formula for the unsteady load is

$$\Delta \hat{c}_p = \Delta C_p^{(0)} + ik\{\Delta C_p^{(1)} + [M_\infty^2/(1 - M_\infty^2)]x\Delta C_p^{(0)}\} \quad (10)$$

where $\Delta C_p^{(0)}$ and $\Delta C_p^{(1)}$ are the steady loads found by solving the steady flow problems associated with the boundary conditions Eqs. (8) and (9). They are computed from the steady potentials as

$$\Delta C_p^{(\ )} = 4\Phi_x^{(\ )}|_{z=0+} \quad (11)$$

Equation (7) yields directly the relation between unsteady and steady AIC's.

$$\hat{a}_{ij} = A_{ij}^{(0)} + ik \left(A_{ij}^{(1)} + \frac{M_\infty^2}{1 - M_\infty^2} x A_{ij}^{(0)} + \frac{1}{M_\infty^2 - 1}\int_{-\infty}^{x} A_{ij}^{(0)} d\xi\right) \quad (12)$$

The coefficients $A_{ij}^{(0)}$ and $A_{ij}^{(1)}$ are steady AIC's of wing panels, but correspond to different unit loads $\Delta C_p^{(0)}$ and $\Delta C_p^{(1)}$. The latter are obtained from the above definition of low frequency AIC's and Eq. (10).

An aerodynamic influence coefficient is defined for unit real and zero imaginary load. Introducing that definition, Eq. (10) yields

$$\Delta C_p^{(0)} = 1 \qquad \Delta C_p^{(1)} = [M_\infty^2/(M_\infty^2 - 1)]x$$

These results show that the steady influence coefficients, $A_{ij}^{(0)}$ and $A_{ij}^{(1)}$, correspond to uniform and to x-linearly varying panel loads, respectively. The two sets of steady influence coefficients are obtained from the integrals

$$A_{ij}^{(0)} = \iint_{S_j} K_s(x_i - \xi, y_i - \eta; M_\infty) d\xi d\eta \quad (13)$$

$$A_{ij}^{(1)} = \frac{M_\infty^2}{M_\infty^2 - 1}\iint_{S_j} \xi K_s(x_i - \xi, y_i - \eta; M_\infty) d\xi d\eta \quad (14)$$

where K_s is the steady kernel function given by Eqs. (1a, b) for $k = 0$.

$A_{ij}^{(0)}$ has been evaluated by Woodward[2] in closed form. $A_{ij}^{(1)}$ was evaluated by D'Sylva[14] by first integrating in closed form in the x-direction, removing the Mangler, Cauchy and logarithmic singularities in the y-direction, and computing the resulting proper integral numerically.

Equation (12) shows that another AIC, termed the integrated downwash, I_{ij}, is needed. It is defined by

$$I_{ij} = \int_{-\infty}^{x} A_{ij}^{(0)} d\xi = \int_{-\infty}^{x} \iint_{S_j} K_s(\bar{x}_i - \xi, y_i - \eta; M_\infty) d\xi d\eta d\bar{x} \quad (15)$$

Performing the outer integration first yields for $M_\infty < 1$

$$I_{ij} = \iint_{S_j} \frac{1}{(y_i - \eta)^2}(x_i - \xi + [(x_i - \xi)^2 + \beta^2(y_i - \eta)^2]^{1/2}) d\xi d\eta \quad (16)$$

The chordwise integral

$$F(\eta) = \int_{x_{LE}(\eta)}^{x_{TE}(\eta)} (x_i - \xi + [(x_i - \xi)^2 + \beta^2(y_i - \eta)^2]^{1/2}) d\xi$$

is solved in closed form. Figure 2 contains all necessary information on the panel geometry and the integration limits.

The spanwise integral of Eq. (16) is a Mangler-type[13] integral. It has been evaluated numerically using the formula

$$\int_{y_L}^{y_R} \frac{F(\eta)}{(y - \eta)^2} d\eta = \int_{y_L}^{y_R} \frac{G(\eta)}{(y - \eta)^2} d\eta + F(y)\frac{y_R - y_L}{(y - y_R)(y - y_L)} + \left.\frac{\partial F(\eta)}{\partial \eta}\right|_y \ln\frac{y - y_R}{y - y_L} \quad (17)$$

$$G(\eta) = F(\eta) - F(y) - \left.\frac{\partial F(\eta)}{\partial \eta}\right|_y (\eta - y)$$

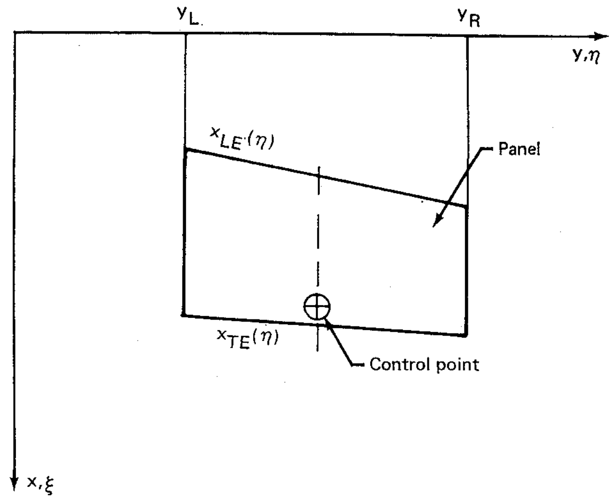

Fig. 2 Panel description.

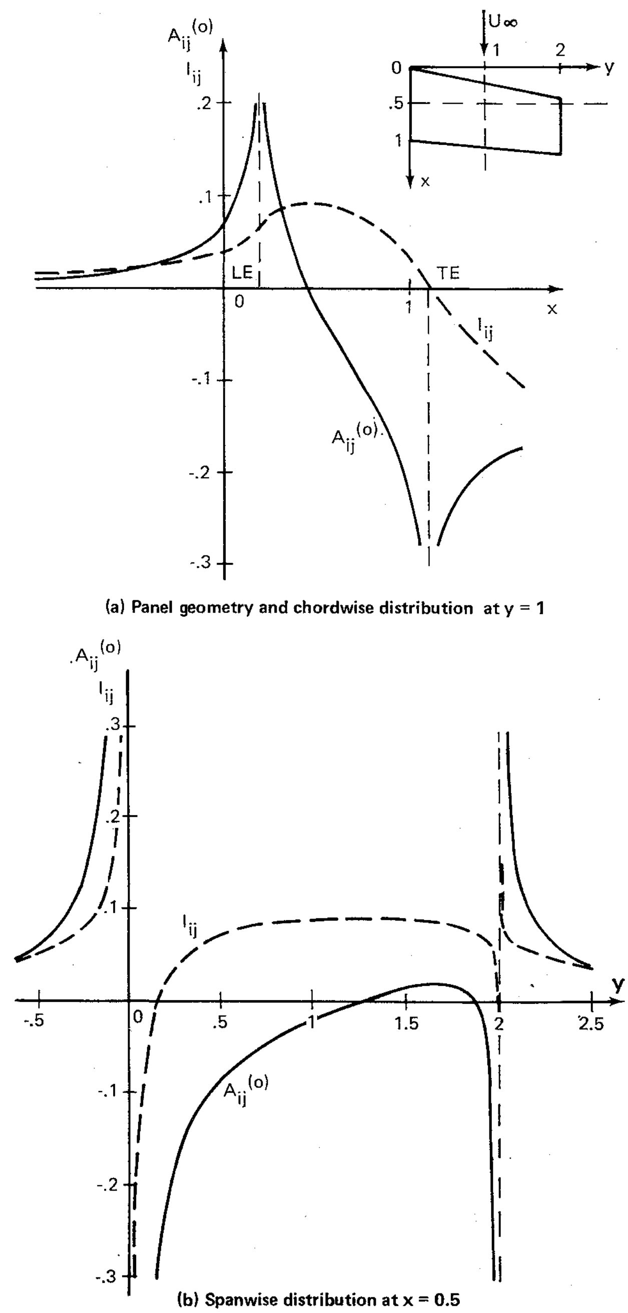

Fig. 3 Example of downwash and integrated downwash of constant load panel in subsonic flow. $M_\infty = 0.5$, $\Delta C_p^{(0)} = 1$.

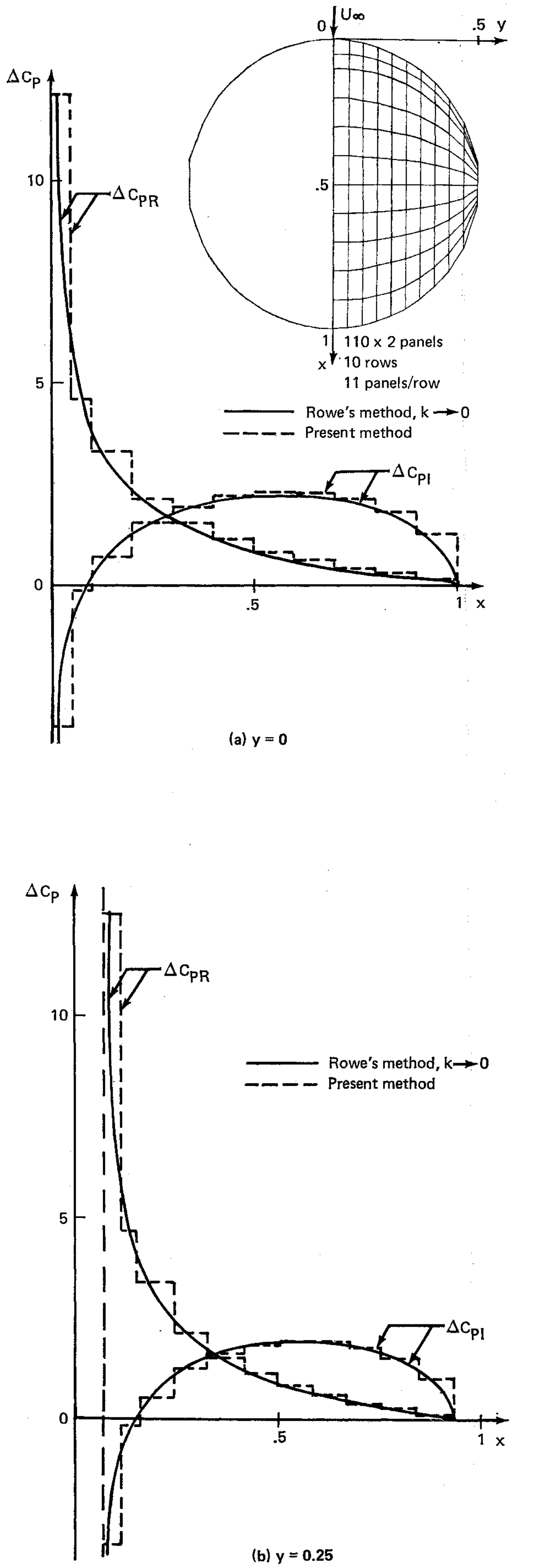

Fig. 4 Circular wing oscillating in pitch about midchord at $M_\infty = 0$; chordwise load distributions.

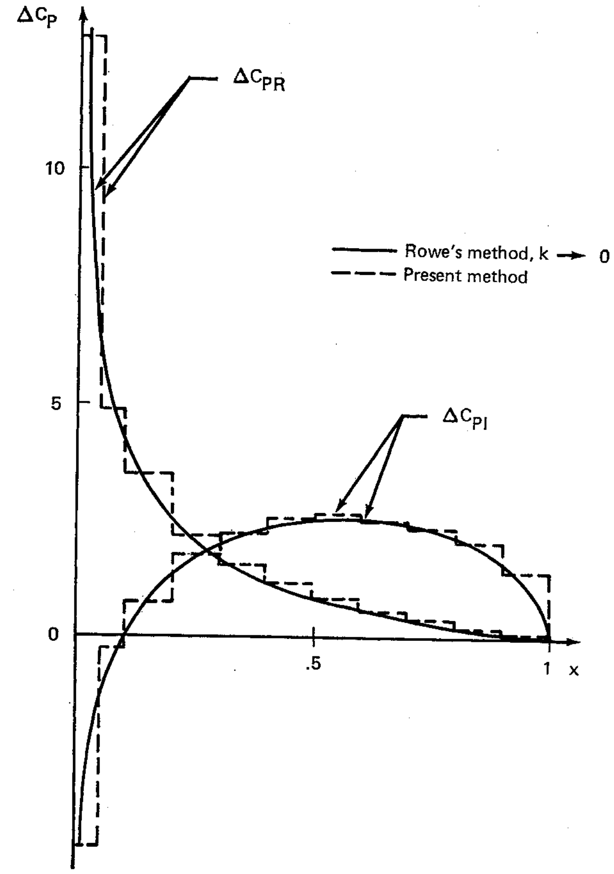

Fig. 5 Circular wing oscillating in pitch about midchord at $M_\infty = 0.5$; chordwise load distribution at $y = 0$.

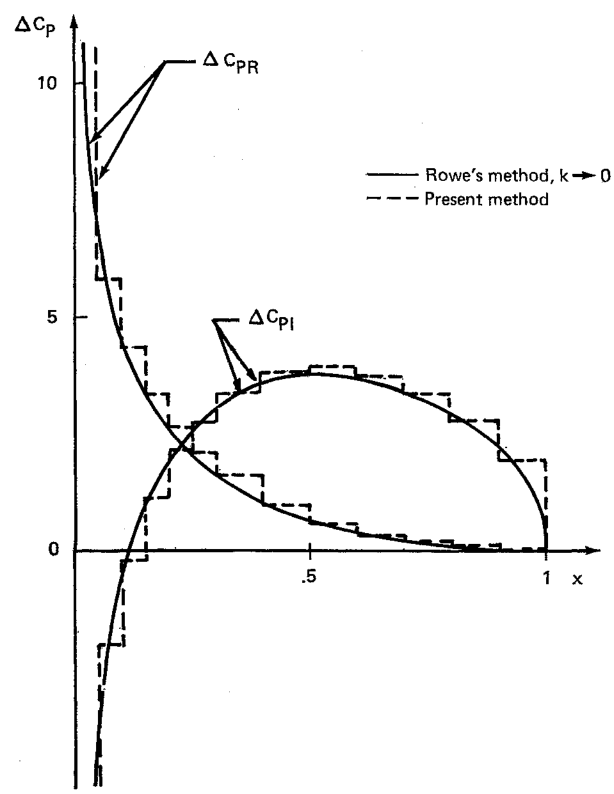

Fig. 6 Circular wing oscillating in pitch about midchord at $M_\infty = 0.9$; chordwise load distribution at $y = 0$.

Figure 3 shows an example for the downwash $A_{ij}^{(0)}$ and integrated downwash I_{ij} across a quadrilateral panel in subsonic flow. The latter is continuous across the leading and trailing panel edges, representing the streamline over a panel of constant aerodynamic load, but is discontinuous across the side edges.

Dynamic Stability Derivatives

A primary application of the low frequency AIC's is in the field of stability and control. Dynamic stability derivatives can be expressed in terms of the described low frequency theory using the method of Etkin[15] which has been reformulated by Rodden and Giesing.[16] The reader is referred to these two references for details of the derivation.

The final equations for the dynamic stability derivatives $c_{L\dot{\alpha}}$ and $c_{M\dot{\alpha}}$ representing the unsteady contributions to lift and pitching moment are

$$\begin{aligned} c_{L\dot{\alpha}} &= (1/S_{ref})\lfloor S_i\rfloor[a_{Rij}]^{-1}[a_{Iij}][a_{Rij}]^{-1}\{1\} \\ c_{M\dot{\alpha}} &= (1/S_{ref}c_{ref})\lfloor S_i(x_0 - x_c)\rfloor[a_{Rij}]^{-1}[a_{Iij}][a_{Rij}]^{-1}\{1\} \end{aligned} \quad (18)$$

The corresponding quasisteady derivatives are found to be

$$\begin{aligned} c_{Lq} &= -(1/S_{ref})\lfloor S_i\rfloor[a_{Rij}]^{-1}\{x_0 - x_c\} \\ c_{Mq} &= -(1/S_{ref}c_{ref})\lfloor S_i(x_0 - x_c)\rfloor[a_{Rij}]^{-1}\{x_0 - x_c\} \end{aligned} \quad (19)$$

where S_{ref} and c_{ref} denote the reference area and length, x_0 is the reference axis of the pitching moment (positive nose up), x_c is the x-location of the panel centroid, and S_i the panel area.

Numerical Results

The method has been used to compute stability derivatives and pressure distributions for a number of wings in subsonic, transonic and supersonic flow. Wings of simple geometry for which analytical data are available have been selected. The computations are based on an extension of Woodward's[2] steady aerodynamic method. Experience with this method has indicated that panel geometry and control point location strongly affect the results. The low frequency unsteady aerodynamic results therefore suffer from similar dependence. The object here however is to demonstrate the validity of the low frequency approximation. The control point was located on the chord line through the centroid of each row of panels at 85% panel chord.

Circular Wing

This wing has been used to evaluate the method in subsonic flow. Chordwise distributions of the pressure jump are shown in Figs. 4, 5, and 6. They are compared with results from Rowe[17] that were obtained executing his computer program for low reduced frequencies. The data agree well for all subsonic Mach numbers. Longitudinal stability derivatives for the incompressible case are listed in Table 1. The derivatives agree reasonably well with those computed by Garner[9] and Spiegel[18] in spite of the simplicity of the present method.

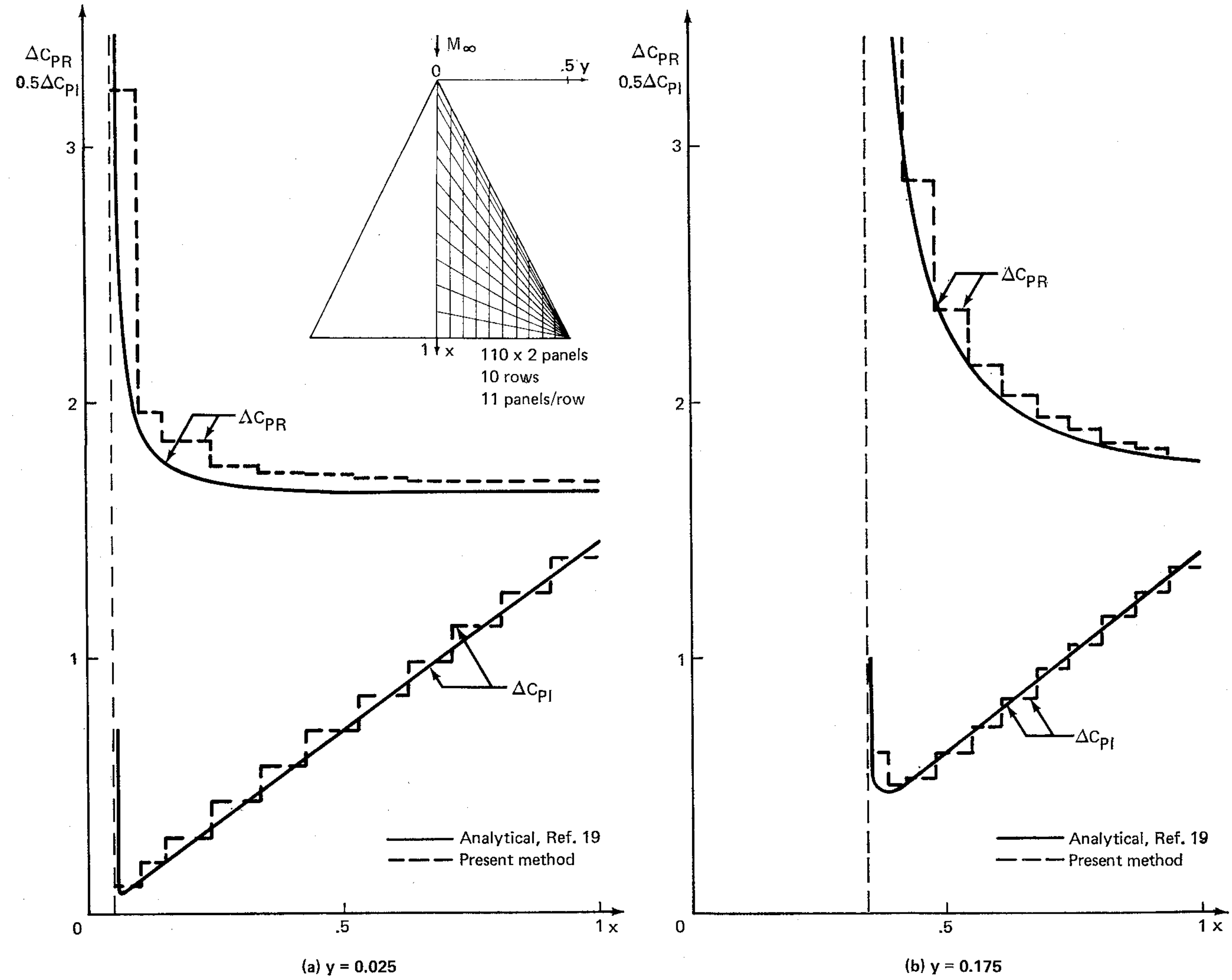

Fig. 7 Slender delta wing ($AR = 2$) oscillating in pitch about the apex at $M = 1.414$, chordwise load distributions.

Table 1 Stability derivatives[a] of circular wing, $M_\infty = 0$

	$c_{L\alpha}$	c_{Lq}	$c_{L\dot{\alpha}} + c_{Lq}$	$c_{M\alpha}$	c_{Mq}	$c_{M\dot{\alpha}} + c_{Mq}$
Spiegel[18]	1.790	...	1.199	0.466	...	−0.135
Garner[9]	1.798	0.470	1.219	0.469	−0.109	−0.122
Present method	1.905	0.502	1.223	0.468	−0.111	−0.156

[a] $S_{ref} = \Pi/4$, $c_{ref} = 1$, $x_0 = 0.5$.

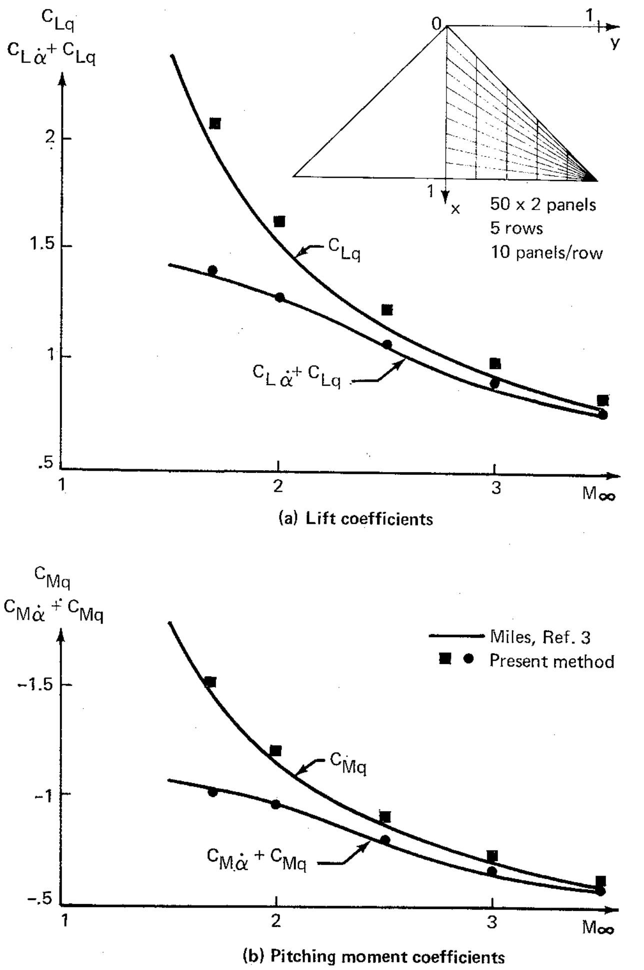

Fig. 8 Wide delta wing ($AR = 4$) in supersonic flow, $S_{ref} = 1$, $c_{ref} = 1$, $x_0 = 0$.

Delta Wings ($AR = 2$ and 4)

Two delta wings have been analyzed in supersonic flow, Figs. 7 and 8. Distribution of the pressure jump is shown in Fig. 7 for a wing that is oscillating in pitch about the apex at Mach 1.414. The results are compared with analytical data from Carafoli.[19] It may be noticed that the imaginary part of the pressure jump is much better predicted than the real part, a fact for which the authors have no satisfactory explanation. This result is also reflected in Table 2 for the corresponding stability derivatives, where the steady type derivatives are less accurate than the unsteady derivatives.

Stability derivatives of the wide delta wing are plotted versus supersonic Mach numbers in Fig. 8. The agreement of dynamic stability derivatives is within 5%.

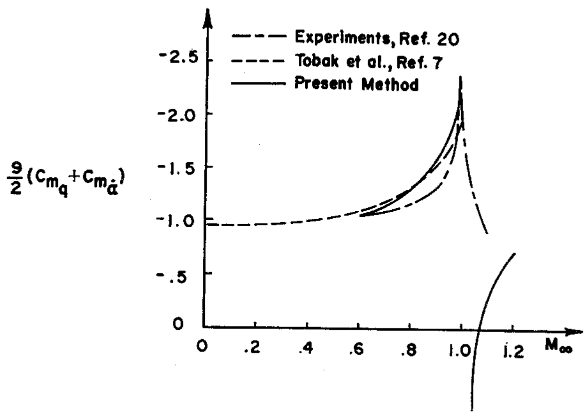

Fig. 9 Damping coefficients for a delta wing ($AR = 2$) in transonic flow.

Damping derivatives $c_{m_{\dot{\alpha}}} + c_{m_q}$ have been calculated for the delta wing of aspect ratio two in the transonic flow regime. The result is compared in Fig. 9 with experimental data of Emerson and Robinson[20] and theoretical data of Tobak and Lessing.[7] Although the linear low frequency theory has questionable validity for freestream Mach numbers close to unity, damping derivatives are predicted quite well in the subsonic regime. The failure of the linear theory in the immediate vicinity of $M_\infty = 1$ is apparent. However, for delta wings of larger aspect ratio ($AR = 3, 4$), the indicated reversal in sign of the damping coefficient at low supersonic Mach numbers has in fact been borne out by experiments.

Rectangular Wing ($AR = 2$)

Section lift coefficients for a pitch oscillation about midchord at Mach 1.732 are compared with data from Nelson[21] in Fig. 10. It should be mentioned that in supersonic flow attention must be paid to the so-called numerical forward feed problem. In using Woodward's approach, panels can influence regions in their forecones if the panel arrangement is not properly chosen. The panel scheme, however, is particularly simple for the rectangular wing and the arrangement shown can be used over a wide Mach number range without introducing errors from forward feed.

Conclusions

A low frequency approximation to the unsteady lifting surface problem has been formulated and applied to planar wings in subsonic and supersonic flow. The method is based on aerodynamic influence coefficients of a slowly oscillating quadrilateral wing panel with constant load amplitude. The unsteady aerodynamic influence coefficients are expressed in terms of equivalent steady coefficients by means of Eq. (12).

Table 2 Stability derivatives[a] of delta wing, $AR = 2$, $M_\infty = 1.414$

	$c_{L\alpha}$	c_{Lq}	$c_{L\dot{\alpha}} + c_{Lq}$	$c_{M\alpha}$	c_{Mq}	$c_{M\dot{\alpha}} + c_{Mq}$
Miles[3]	2.596	2.060	1.845	−1.730	−1.544	−1.383
Present method	2.652	2.132	1.842	−1.748	−1.581	−1.377

[a] $S_{ref} = 1$, $c_{ref} = 1$, $x_0 = 0$.

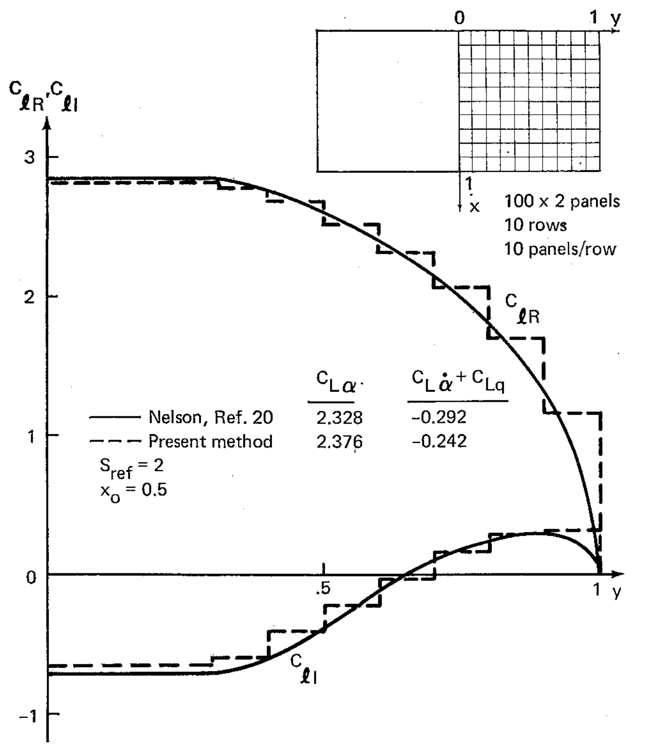

Fig. 10 Rectangular wing oscillating in pitch about midchord at $M_\infty = 1.732$, section lift coefficient.

The method is an extension of the well known Woodward scheme in steady aerodynamics, and, although it has only been demonstrated for planar wings, it can easily be extended to nonplanar wings including wing-body combinations.

The method is limited to reduced frequencies of the airplane's oscillations that are small compared to unity. Its main application therefore is expected to be in stability and control analysis for large aircraft whose characteristic motions satisfy the restriction on reduced frequency. The method includes the first order effects of an unsteady wake and of finite speeds of disturbance propagation; it is clearly distinguished from a quasisteady approach. The method, therefore, is suitable for predicting dynamic stability derivatives.

The computed pressure distributions and dynamic stability derivatives compare reasonably well with analytical data and those from numerical unsteady lifting surface programs. The numerical values which are presented are intermediate results obtained from a computer program being developed for application to arbitrary wing-body combinations in subsonic and supersonic unsteady flow. The results appear to validate the method for the thin lifting surface part of that more general problem.

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